Inequality with multiplicative nested radicals.

https://www.linkedin.com/feed/update/urn:li:activity:6531200184045703168

Prove the inequality

$$\sqrt{2\sqrt[3]{3\sqrt[4]{4\sqrt[5]{\ldots\sqrt[n]{n}}}}} < 2, n \geq 2.$$

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Let $r_n := \sqrt{2\sqrt[3]{3\sqrt[4]{4\sqrt[5]{\ldots\sqrt[n]{n}}}}}$, $n \geq 2$.

Since $n! \geq 2 \cdot 3^{n-2}$, $n \geq 2$ and $\max\limits_{n\in\mathbb{N}} n^{\frac{1}{n}} = 3^{\frac{1}{3}}$ then for $k \geq 3$ holds

$k^{\frac{1}{k!}} = \left(k^{\frac{1}{k}}\right)^{\frac{1}{(k-1)!}} \leq 3^{\frac{1}{3(k-1)!}} \leq 3^{\frac{1}{2\cdot 3^{k-2}}}$ and, therefore,

$$r_n = 2^{\frac{1}{2!}} \cdot 3^{\frac{1}{3!}} \cdot \ldots \cdot n^{\frac{1}{n!}} \leq 2^{\frac{1}{2}} \cdot 3^{\frac{1}{2\cdot 3}} \cdot \ldots \cdot 3^{\frac{1}{2\cdot 3^{n-2}}} <$$

$$\sqrt{2 \cdot 3^{\frac{1}{3}+\frac{1}{3^2}+\ldots+\frac{1}{3^{n-2}}+\ldots}} = \sqrt{2 \cdot 3^{\frac{1}{2}}} = \sqrt[4]{12} < 2.$$